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Phase structure of matrix models through orthogonal polynomials

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Abstract. In the non-perturbative regime, matrix models display a large-N phase transition. For finite but large N, the transition is anticipated by strong oscillations in some coefficients in the recurrence relations for the orthogonal polynomials that allow the calculation of the partition function. This paper shows how to perform the limit, requiring the definition of different interpolating functions according to the parity of polynomials, in the cases of a single or two interacting matrices.

1. Introduction

In recent times there has been a rising interest in the study of the large-N, or planar, limit of matrix models. The simplest case of a Gaussian partition function was considered by Wigner; the statistical properties of the eigenvalues of such matrices closely match those of highly energetic sets of nuclear levels. The next step, the introduction of a cubic or quartic interaction, was then performed by Brézin *et al* in a remarkable paper [1]. These interacting models, besides their importance in the investigation of the effectiveness of the planar limit as a tool for performing non-perturbative approximations in field theories, may describe the statistics of random bidimensional surfaces [2]. The more difficult case of two or more interacting matrices was solved by Mehta [3] and was subsequently shown to be equivalent to an Ising model on planar graphs [4].

Models with unitary matrices are extensively studied since they follow from the factorisation of Yang-Mills theories in two-dimensional lattices, as a consequence of gauge invariance. There are various methods for performing the large-N limit of the single matrix models, all sharing a $SU(N)\times SU(N)$ invariant action: saddle-point equations, collective fields, orthogonal polynomials and perhaps others. The orthogonal polynomial method seems to be the most effective, since it allows the systematic calculation of the various orders in the $1/N$ expansion and extends also to the case with two matrices.

The large-N limit of matrix models shows the interesting feature of one or more phase transitions, as explained by Shimamune [5], Cicuta *et al* [6, 7]† and Gross and Witten [8].

The aim of this paper is to generalise the results of Bessis [9] and Mehta [3] on orthogonal polynomials, by showing how the phase transition develops in their scheme,

† Cicuta *et al* were unaware of the existence of [5] when submitting [6] for publication.

once the range of the parameters is extended beyond the transition critical borders. The main difference that arises when considering the enlarged region of parameters is the appearance of a strong oscillatory character of certain coefficients, depending on their parity. This compels one to perform the large-N limit with a number of interpolating functions which is doubled with respect to the previously known cases. The new solutions are quite different and no longer describe the perturbative regime; they may be of interest in the study of the related statistical models.

2. The single matrix model

Starting from the partition function for a $N\times N$ Hermitian matrix field

$$\begin{aligned} Z_n &= \int \mathrm{d}A \exp\left(\frac{-m^2}{2}\operatorname{Tr}A^2 - \frac{g}{N}\operatorname{Tr}A^4\right)\left[\int \mathrm{d}A\exp\left(-\frac{|m^2|}{2}\operatorname{Tr}A^2\right)\right]^{-1} \\ &= \int \prod_{i=1}^{N} \mathrm{d}\lambda_i\, \Delta(\lambda_1,\ldots,\lambda_N)^2 \exp\left[-\sum_{i=1}^{N}\left(\frac{m^2}{2}\lambda_i^2 + \frac{g}{N}\lambda_i^4\right)\right] \\ &\quad\times\left[\int \prod_{i=1}^{N}\mathrm{d}\lambda_i\,\Delta(\lambda_1,\ldots,\lambda_N)^2\exp\left(-\sum_{i=1}^{N}\frac{|m^2|}{2}\lambda_i^2\right)\right]^{-1} \end{aligned} \tag{2.1}$$

with $\Delta(\lambda_1,\ldots,\lambda_N)=\Pi_{i<j}(\lambda_i-\lambda_j)$, Bessis [9] has shown an interesting method for deriving the $1/N$ expansion. A detailed technical review of the method is given in [10].

The Δ function is a Vandermonde determinant, therefore its value is not changed if each column of the matrix $\Delta_{ij}=\lambda_i^{j-1}$ is replaced by a linear combination of the others. He exploited this fact by introducing a class of orthogonal polynomials $P_k(\lambda)$ such that

$$\int_{-\infty}^{+\infty} \mathrm{d}\lambda\, P_i(\lambda)P_j(\lambda)\exp\left[-\left(\frac{m^2}{2}\lambda^2+\frac{g}{N}\lambda^4\right)\right] = h_i\delta_{ij} \tag{2.2}$$

with $P_0(\lambda)=1$, $P_1(\lambda)=\lambda$ and $P_k(-\lambda)=(-1)^kP_k(\lambda)$ due to the parity of the potential. These polynomials satisfy the recursion relation

$$P_{k+1}(\lambda)=\lambda P_k(\lambda)-R_kP_{k-1}(\lambda). \tag{2.3}$$

For the coefficients R_k one derives the equations

$$k=R_k[m^2+(4g/N)(R_{k+1}+R_k+R_{k-1})] \tag{2.4}$$

$$R_k=h_k/h_{k-1}. \tag{2.5}$$

With this trick, the integrations may be performed, since the different eigenvalues become decoupled, and one obtains:

$$Z_N=\prod_{i=0}^{N-1}\frac{h_i}{\hat{h}_i}=\left(\frac{h_0}{\hat{h}_0}\right)^N\prod_{k=1}^{N-1}\left(\frac{R_k}{\hat{R}_k}\right)^{N-k} \tag{2.6}$$

where $\hat{h}_0=h_0(|m^2|, g=0)$ and so on.

The case $m^2>0$ has been solved by Bessis [9] in the planar limit ($N\to\infty$) and also the two subsequent orders have been calculated. In the large-N limit, by setting $R(k/N)=(1/N)R_k$, $k/N=x\in[0,1]$ for $0\leqslant k\leqslant N$, one gets from (2.4) the equation

$$x=R(x)[m^2+12gR(x)] \qquad R(0)=0. \tag{2.7}$$

The vacuum energy

$$E = -\lim_{N\to\infty} \frac{1}{N^2} \log Z_N = -\lim_{N\to\infty} \frac{1}{N} \log\left(\frac{h_0}{\hat{h}_0}\right) - \int_0^1 (1-x) \log \frac{R(x)}{\hat{R}(x)} \, dx \tag{2.8}$$

reproduces the well known result obtained by Brezin *et al* [1].

The case $m^2 < 0$ leads to quite different solutions. One may easily compute the first few coefficients R_k and find out that, in the limit $N \to \infty$, they behave as different powers of N according to k being even or odd. This implies two different boundary conditions on the solution of the continuum version of the recursion relation.

More precisely, one finds by direct computation that

$$\begin{aligned} R_{2k} &= \frac{2k}{|m^2|}\left[1 + \frac{2k}{N}\left(\frac{4g}{m^4}\right) + \ldots\right] \\ R_{2k+1} &= N\frac{|m^2|}{4g}\left[1 - \frac{2k+1}{N}\left(\frac{4g}{m^4}\right) + \ldots\right]. \end{aligned} \tag{2.9}$$

The correct procedure is to define two different limiting functions

$$R_{2k} \to \frac{N|m^2|}{4g} r\left(\frac{4gx}{m^4}\right) \qquad R_{2k+1} \to \frac{N|m^2|}{4g} \rho\left(\frac{4gx}{m^4}\right) \tag{2.10}$$

with r and ρ satisfying the coupled equations that follow from (2.4) with k even and k odd, $(t = (4g/m^4)x)$:

$$\begin{aligned} t &= r(t)(m^2/|m^2| + 2\rho(t) + r(t)) \\ t &= \rho(t)(m^2/|m^2| + 2r(t) + \rho(t)). \end{aligned} \tag{2.11}$$

A first solution is $r(t) = \rho(t) = \frac{1}{6}(-m^2/|m^2| + \sqrt{1+12t})$ which is that previously found by Bessis [9] and solves (2.7). The second and more interesting solution of the system is given by the pair

$$\begin{aligned} r(t) &= \tfrac{1}{2}(-m^2/|m^2| - \sqrt{1-4t}) \\ \rho(t) &= \tfrac{1}{2}(-m^2/|m^2| + \sqrt{1-4t}) \end{aligned} \tag{2.12}$$

which is admissible only for $m^2 g^{-1/2} < -4$ and displays the correct behaviour at $t = 0$. The critical value coincides with the one previously found by Shimamune [5] and Cicuta *et al* [6] and leads to a new phase for the planar ϕ^4 model in zero dimensions.

3. The two-matrices planar model

The much more difficult case with two interacting Hermitian matrices was then considered by Itzykson and Zuber [11] and the final solution was elegantly found by Mehta [3]. By relating the problem of calculating the partition function

$$Z_N = \int dA \, dB \exp[-m^2 \operatorname{Tr}(A^2 + B^2) - (g/N) \operatorname{Tr}(A^4 + B^4) + 2c \operatorname{Tr}(AB)] \tag{3.1}$$

to a Cauchy problem for the heat equation, Mehta simplified the previous integral into

$$Z_N = C_N \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \, \mathrm{d}\mu_i \, \Delta(\lambda_1, \ldots, \lambda_N)\Delta(\mu_1, \ldots, \mu_N)$$

$$\times \exp\left(-\sum_{i=1}^{N} [m^2(\lambda_i^2+\mu_i^2)+(g/N)(\lambda_i^4+\mu_i^4)-2c\lambda_i\mu_i]\right) \tag{3.2}$$

$$C_N = \left(\frac{\pi}{\sqrt{2c}}\right)^{N(N-1)} \left(\prod_{i=1}^{N} i!\right)^{-1} \tag{3.3}$$

thus overcoming the impossibility of a simultaneous diagonalisation of the two matrix variables.

This result, though simple-looking, is far from being trivial; the Δ functions appear with exponent one and the limit $c \to 0$ requires great care, though giving immediate factorisation in (3.1).

Without loss of generality, c is taken non-negative.

The next step is to define a set of polynomials $P_k(\lambda) = \lambda^k + \mathrm{O}(\lambda^{k-2})$ with definite parity, and satisfying the orthogonality property

$$\int_{-\infty}^{+\infty} \mathrm{d}\lambda \int_{-\infty}^{+\infty} \mathrm{d}\mu \, P_i(\lambda)P_j(\mu) \exp[-m^2(\lambda^2+\mu^2)-(g/N)(\lambda^4+\mu^4)+2c\lambda\mu] = h_i\delta_{ij}. \tag{3.4}$$

These polynomials may be chosen according to the recurrence relation

$$P_{k+1}(\lambda) = \lambda P_k(\lambda) - R_k P_{k-1}(\lambda) - S_k P_{k-3}(\lambda) \tag{3.5}$$

with coefficients given by

$$\begin{aligned} &f_i[m^2+(2g/N)(R_{i-1}+R_i+R_{i+1})] = cR_i \\ &cf_i = -\tfrac{1}{2}i + R_i[m^2+(2g/N)(R_{i-1}+R_i+R_{i+1})]+(2g/N)(S_i+S_{i+1}+S_{i+2}) \\ &S_i = (2g/Nc)f_i f_{i-1} f_{i-2} \end{aligned} \tag{3.6}$$

having set $h_i/h_{i-1} = f_i$.

By replacing $\Delta_{ij} = \lambda_i^{j-1}$ with $\Delta_{ij} = P_{j-1}(\lambda_i)$, sharing the same determinant $\Delta(\lambda_1, \ldots, \lambda_N)$, one easily finds

$$Z_N = N!\, C_N h_0^N \prod_{k=1}^{N-1} f_k^{N-k}. \tag{3.7}$$

When $m^2 < c$, the potential $\omega(\lambda, \mu) = m^2(\lambda^2+\mu^2)+(g/N)(\lambda^4+\mu^4)-2c\lambda\mu$ develops a minimum different from the origin. In this case, the computation by saddle-point methods of the first few coefficients f_k shows a dual behaviour: for k even, f_k is of order one, while for k odd, the coefficient is of order N. One therefore defines two different functions $F(x)$ and $\Phi(x)$, taking the values respectively $(1/N)f_2 = 0$ and $(1/N)f_1 = (c-m^2)/2g$ at $z=0$. For $m^2 > c$, the two functions should coincide and take the value zero at $x=0$. Analogously, one defines two functions $\rho(x)$ and $r(x)$ associated with coefficients R_{2i+1} and R_{2i}.

Elimination of S_i, and some algebra, give in the large-N limit the following equations:

$$
\begin{aligned}
r &= F\frac{m^2(c+2g\Phi)}{c^2-2cg(F+\Phi)-12g^2F\Phi} \\
\rho &= \Phi\frac{m^2(c+2gF)}{c^2-2cg(F+\Phi)-12g^2F\Phi}
\end{aligned}
\tag{3.8}
$$

together with

$$
\begin{gathered}
(F-\Phi)(c^2-4g^2F\Phi)\{[c^2-2cg(F+\Phi)-12g^2F\Phi]^2-c^2m^4\}=0 \\
c(F+\Phi)+x-\frac{12g^2}{c}F\Phi(F-\Phi)=cm^4\left(\frac{c^2(F+\Phi)+8gcF\Phi+4g^2F\Phi(F+\Phi)}{[c^2-2cg(F+\Phi)-12g^2F\Phi]^2}\right).
\end{gathered}
\tag{3.9}
$$

Therefore, the following three cases are possible:

$$F=\Phi \tag{3.10a}$$

$$F\Phi=c^2/4g^2 \tag{3.10b}$$

$$[c^2-2gc(F+\Phi)-12g^2F\Phi]^2-c^2m^4=0. \tag{3.10c}$$

The first case (3.10a) gives Mehta's solution, the function $F(x)$ satisfying a fifth-order algebraic equation:

$$c^2F(x)[1-m^4/(c-6gF)^2]-12g^2F(x)^3=-\tfrac{1}{2}cx. \tag{3.11}$$

In the limit $c\to 0$ one obtains

$$
\begin{aligned}
F(x) &= \frac{m^4c}{72g^2x}\left[\left(1+\frac{12g}{m^4}x\right)^{1/2}-1\right]^2 \\
r(x) &= \frac{m^2}{12g}\left[\left(1+\frac{12g}{m^4}x\right)^{-1/2}-1\right]
\end{aligned}
\tag{3.12}
$$

which reproduce the single matrix results, once the factor $\frac{1}{2}$ is taken into account in the mass term.

In the second case (3.10b), the condition that $F(x)$ and $\Phi(x)$ should be real for any x in $[0, 1]$ is satisfied only in the limit $c\to 0$, and gives

$$
\begin{aligned}
F(x) &= (c/8g^2x)[(m^4-4gx)^{1/2}+|m^2|]^2 \\
\Phi(x) &= (c/8g^2x)[(m^4-4gx)^{1/2}-|m^2|]^2.
\end{aligned}
\tag{3.13}
$$

In the computation of the vacuum energy, the vanishing quantity c cancels with the normalisation constant. One finds the condition $m^2\leqslant -2\sqrt{g}$, which corresponds to the critical point found in the single matrix case.

The third case (3.10c) is more interesting. The equations for the computation of $F(x)$ and $\Phi(x)$ are:

$$
\begin{gathered}
[c^2-2gc(F+\Phi)-12g^2F\Phi]^2=m^4c^2 \\
-x+8gF\Phi+(16g^2/c)F\Phi(F+\Phi)=0.
\end{gathered}
\tag{3.14}
$$

The system has in particular the following solutions:

$$
\begin{aligned}
F(x)\Phi(x) &= (c/24g^2)\{(2c-m^2)-[(2c-m^2)^2-6xg]^{1/2}\} \\
F(x)+\Phi(x) &= (1/4g)\{-m^2+[(2c-m^2)^2-6xg]^{1/2}\}.
\end{aligned}
\tag{3.15}
$$

At $x=0$, one correctly has $F(0)=0$ and $\Phi(0)=(c-m^2)/2g$.

The condition for reality of $F(x)$ and $\Phi(x)$ for all $x \in [0, 1]$, that is $(F+\Phi)^2 - 4F\Phi > 0$, gives the boundary separating the two phases, cases (3.10a) and (3.10c), in the plane of the dimensionless parameters $cg^{-1/2} > 0$ and $-\infty < m^2 g^{-1/2} < +\infty$.

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